Robust Dynamic Asset Allocation with Model Misspecification

Esben Hedegaard*

November 12, 2014

Abstract

I derive the robust dynamic trading strategy when trading is costly and returns are predictable by signals with different mean-reversion speeds, but the alpha-decay of the signals is misspecified. The robust strategy puts less weight on signals for which misspecification is more important, and as a result the robust strategy down-weights highly persistent signals and highly volatile signals relative to less persistent and less volatility signals. In out-of-sample tests, the robust strategy outperforms the non-robust optimal strategy in scenarios with high transaction costs.

*esben.hedegaard@stern.nyu.edu
1 Introduction

In trying to generate abnormal returns, active managers try to forecast returns and profit from their predictions. Since trading is costly, they face a tradeoff between trading aggressively on their signals and minimizing transaction costs. If the predictive model is imperfect, standard optimization methods put too much weight on the signals and as a result the investor trades too much and incurs too high transaction costs. Systematic strategies such as momentum and short-term reversals have high turnover and the profitability of these strategies depend crucially on minimizing transaction costs.

I consider an investor who predicts returns using a set of return-predicting factors. In the absence of transaction costs, the investor would each period trade to the optimal myopic Markowitz portfolio. Since the return-predicting factors change over time, the Markowitz portfolio is a moving target in the language of Gărleanu and Pedersen (2013), but due to transaction costs the investor does not trade all the way to the myopic Markowitz portfolio. Without regards for misspecification, Gărleanu and Pedersen (2013) show that the optimal trading strategy is based on two principles: (1) aim in front of the target, and (2) trade partially toward the aim. The aim portfolio is a weighted average of the current and future Markowitz portfolios and depends on the mean-reversion speed (alpha decay) of the predictors. Since the investor can enjoy high returns for a longer time when the predictor is persistent, the investor trades more aggressively on these signals. Conversely, if a high expected return is predicted to disappear quickly, the investor does not spend money on building up a large position in this asset.

If the evolution of the predictors is misspecified, the investor systematically aims toward a sub-optimal target. I solve for the optimal trading strategy when the model is misspecified, and the resulting robust trading strategy works well for a range of alternative dynamics of the predictors. More precisely, robust control theory introduces a ‘model confidence region,’ similar to a standard parameter confidence interval, of alternative models that are close the the estimated model. The investor worries about some worst-case model within this model confidence region and derives the optimal trading strategy under this worst-case model. I demonstrate that the principles for the optimal non-robust strategy carry over to the robust strategy, but that the target portfolios are changed. The robust strategy puts less weight on signals where the investor is hurt the most if the dynamics of the signals are misspecified. This leads to two principles for the robust strategy: (1) trade less on highly persistent signals, and (2) trade less on highly volatile signals.

The persistent signals are ‘good’ signals in the sense that the investor expects to enjoy a
high expected return for a long time when receiving a signal from a persistent predictor. As discussed above, the investor trades more aggressively on persistent signals for this reason. However, should the signal turn out to be less persistent than expected, it will hurt the investor greatly. The investor has incurred transaction costs building up a position and he now has a large risk from the non-zero position. Should the signal disappear more quickly than expected, the investor is exposed to risk and must incur transaction costs to close out the position again. The robust investor down-weights the persistent signals relative to the less persistent signals, but of course still trades more aggressively on more persistent signals.

The second effect of robustness is that the robust investor trades less on volatile signals. Two signals may have the same mean-reversion speed, but if one is more volatile there is more uncertainty about the future of this signal. As a result, the robust investor puts less weight on volatile signals compared to signals that have a more certain future. This is in contrast to the non-robust strategy, for which the optimal position only depends on the mean-reversion speeds of the signals.

An investor who seeks robustness must choose the degree of robustness, which determines the size of the model confidence region. With only a small desire for robustness he simply follows a strategy close to the non-robust optimal strategy. With a high desire for robustness he follows the optimal strategy for a model that he might consider very unrealistic. As the desire for robustness is increased, the investor acts more conservatively and the volatility of the portfolio decreases. I calibrate the desire for robustness to target a specific level of volatility, which provides a transparent and intuitive way of choosing the desire for robustness. The investor believes that returns are predictable, but he also does not fully trust his model. As a result, in my empirical application I assume he is only willing to engage in a strategy with a certain level of volatility.

The main feature of the model in this paper and in Gărleanu and Pedersen (2013) is the combination of alpha-decay and transaction costs. To empirically test the robust dynamic model, I therefore focus on a strategy with known out-of-sample performance and high turnover; momentum. I apply the method to momentum strategies across four different asset classes from 1980 to 2013. The four asset classes are the 10 industry portfolios, 18 international equity indices, 24 commodity futures, and 8 foreign exchange rate pairs.1

---

1Momentum as been documented in industry indices by Moskowitz and Grinblatt (1999), in country indices by Asness et al. (1997) and Bhojraj and Swaminathan (2006), in commodity futures returns by Harvey and Erb (2006), Gorton and Rouwenhorst (2006), and Gorton et al. (2013), and in currencies by LeBaron (1999) and Okunev and White (2003). Asness et al. (2013) show that returns to momentum strategies are positively correlated across asset classes and negatively correlated with returns to value strategies.
While most existing research form equal- or rank-weighted long/short portfolios which do not use the correlation structure of asset returns and do not account for transactions costs, I document that mean-variance optimization greatly improves the out-of-sample performance of momentum strategies over standard equal-weighted strategies. Of course, the daily turnover for this strategy is tremendous, which makes it impossible to implement directly. The dynamic trading problem glues together the single-period myopic Markowitz into a dynamic problem which balances the tradeoff between trading to the optimal myopic Markowitz portfolio and minimizing transaction costs.

I compare the out-of-sample performance of the robust trading strategy to the performance of the corresponding optimal trading strategy in which the investor’s risk aversion has been chosen to ensure the same level of volatility as for the robust strategy. As the level of transaction costs decreases, both the non-robust and robust trading strategies converge to the myopic Markowitz strategy. However, the robust strategy performs better than the optimal strategy when transaction costs are high. Of course, it is precisely when transaction costs are high that model misspecification will be most damaging to the investor and consequently where the robust strategy provides the greatest advantage.

These results demonstrate that there is momentum in exchange rates. While the standard rank-weighted momentum strategy fails to generate significant alpha for foreign exchange rates, the robust momentum strategy generates an alpha which is significant at the 5% level or better, depending on the level of transaction costs. Thus, past returns on foreign exchange rates do predict future returns.

There are several attractive features of the robust dynamic trading problem presented in this paper. First, robust control integrates the estimation and decision making process, which are typically dealt with separately, in a dynamic model with transaction costs. Second, it is built on a solid axiomatic foundation (Gilboa and Schmeidler, 1989). Third, the problem considered here gives a new insight into the effect of robustness, and we can intuitively understand how the worst-case model differs from the benchmark model. Fourth, the model performs well out-of-sample.
2 Literature Review

There is a large literature which considers investment decisions by a Bayesian investor under model uncertainty. This paper is more closely related to a relatively small literature that considers portfolio choice when the investor has a desire for robustness. Maenhout (2004) models an investor maximizing his life-time utility from consumption of a single good, while investing in one risk-less asset and one risky asset. The risky asset follows a geometric Brownian motion with drift, and the investor seeks robustness wrt. the dynamics of the risky asset. Maenhout (2004) shows that robustness reduces the optimal equity weight in the portfolio, but increases the relative importance of the intertemporal hedging demand. The portfolio choice of a robust investor is observationally equivalent to the portfolio choice of an investor with higher risk aversion and helps explain the equity premium puzzle. This paper differs in several ways. I consider the portfolio decision when the investor can invest in many assets, expected returns have alpha-decay, and the investor faces transaction costs. Interestingly, in my model, a desire for robustness is not observationally equivalent to higher risk aversion, and a desire for robustness leads to better out-of-sample performance than higher risk-aversion when transaction costs are high.

This paper is also related to Garlappi et al. (2007) who considers a single-period optimization problem when the investor has a desire for robustness. They demonstrate how a desire for robustness leads to a higher out-of-sample Sharpe ratio than that of mean-variance portfolios and Bayesian portfolios. The model Garlappi et al. (2007) is a single period model in which the investor seeks robustness wrt. the current expected return and does not incur transaction costs. In contrast, this paper considers a dynamic model with transaction costs in which the investor seeks robustness wrt. the evolution of future expected returns.

The benchmark model in this paper, i.e., the model that the investor distrusts and seeks robustness for, is the model in Gârleanu and Pedersen (2013) who extend Engle and Ferstenberg (2007) to a setting with predictable returns. Like Heaton and Lucas (1996), Grinold (2006), Engle and Ferstenberg (2007), and Gârleanu and Pedersen (2013) I model transaction costs as quadratic in the transaction size. Grinold (2006) derives the optimal steady-state position when returns are predictable by a single predictor per security and transaction costs are quadratic. Optimal execution strategies, in which the new portfolio is exogenously given, is studied in Almgren and Chriss (2000), and Engle and Ferstenberg

---

Robust decision making under model uncertainty was pioneered in economics by Hansen and Sargent (1995) and Hansen et al. (2006).\textsuperscript{3} Hansen et al. (2006) also establish the connection between robust control and the max-min expected utility of Gilboa and Schmeidler (1989). Hansen et al. (2006) view robust control theory as a way to specify the set of models in the theory of Gilboa and Schmeidler (1989) by taking as benchmark a single model that is regarded as the decision maker’s approximating model, and surrounding it with a cloud of models that are close as measured by relative entropy. Anderson et al. (2003) study the link between the approximating model, model misspecification, asset prices, and model detection statistics.

3 Model

3.1 The Benchmark Dynamic Model

The starting point for this paper is the model presented in Gârleanu and Pedersen (2013), which encompasses Engle and Ferstenberg (2007) as a special case. Following Engle and Ferstenberg (2007) and Gârleanu and Pedersen (2013) I formulate the problem in terms of dollar returns instead of the more common percentage returns, allowing for tractable modeling of transaction costs.

Consider a market with $n$ securities with prices given by the vector $p_t$. The securities’ price changes between time $t$ and $t+1$ (not returns) in excess of the risk-free rate are collected in a vector $r_{t+1} = p_{t+1} - p_t$ given by

$$r_{t+1} = \mu_t + u_{t+1}$$

where $\mu_t$ is the expected return known to the investor at time $t$, and $u_{t+1}$ is an iid. noise term with mean zero and variance $V(u_{t+1}) = \Sigma$. The expected return $\mu_t$ is determined from $m$ mean-reverting signals collected in the vector $f$, such that

$$\mu_t = G f_t$$

$$f_{t+1} = (I - \Phi)f_t + C\varepsilon_{t+1},$$

\textsuperscript{3}Hansen and Sargent (1995) derive time-invariant decision rules to a discounted Gaussian linear regulator problem with risk-sensitivity, and Hansen et al. (2006) establish the link between risk-sensitivity and robust control.
where $f$ is a $m \times 1$ vector of factors that predict returns, $G$ is a $n \times m$ matrix of factor loadings, $\Phi$ is a $m \times m$ positive-definite matrix of mean-reversion coefficients for the factors, and $V(\varepsilon_{t+1}) = I$. (3) implies that the factors, and hence the expected returns, mean-revert to zero. An intercept can be accommodated with a constant factor $f_1 \equiv 1$. The variance of the innovation-term is $V(C\varepsilon_{t+1}) = \Omega$.

The investor is allowed to trade at each point in time, $t = 0, 1, \ldots, T$, but incurs a cost of trading. This cost equals the number of shares traded times the price impact of the trade; $TC_t = \Delta x'_t \Delta p_t$, where $x_t$ is the number of shares held. The total price impact of a trade is taken to be linear in trade size, as in Almgren and Chriss (2000): $\Delta p_t = \Lambda \Delta x_t$, such that the transaction costs are quadratic in trade size:

$$TC = \sum_{t=1}^{T} \Delta x'_t \Lambda \Delta x_t.$$  \hfill (4)

Huberman and Stanzl (2004) show that when the price impact of trades is permanent and time-independent, only linear price-impact functions rule out so-called quasi arbitrage. For this reason, the assumption of quadratic transaction costs should be seen as a strength of the model, not a weakness.

Let $\gamma$ denote the investor’s coefficient of absolute risk aversion. To ease the application of dynamic programming techniques, the problem is formulated as an infinite-horizon problem. The investor’s problem is to choose a trading strategy $x_t, t = 0, 1, 2, \ldots$ to maximize the present value of future expected excess returns, penalized for risk and transaction costs:

$$\max_{(x_t)_{t=0}^{\infty}} E_0 \left( \sum_{t=0}^{\infty} \beta^t \left( x'_t \mu_t - \frac{\gamma}{2} x'_t \Sigma x_t - \frac{1}{2} \Delta x'_t \Lambda \Delta x_t \right) \right)$$  \hfill (5)

subject to

$$\mu_t = G f_t$$  \hfill (6)

$$f_{t+1} = (I - \Phi) f_t + C \varepsilon_{t+1}$$  \hfill (7)

where $\beta$ is the discount factor. The scaling by $\frac{1}{2}$ is included to keep the notation consistent with Gårleanu and Pedersen (2013). The problem assumes that the return innovations are conditionally homoscedastic with covariance matrix $\Sigma$, but I relax this assumption in the implementation in Section 6. There is an implicit wealth constraint in the problem: Since the investor maximizes the present value of future expected excess returns, it is implicitly
assumed that he can borrow and lend at the risk-free rate.

The model captures in a stylized way how active investors try to beat the market by forecasting future returns and profit from these predictions. They face a tradeoff between the expected return to trading and the risk and cost of trading. Executing a large trade quickly will result in large price pressure, but little variation in the execution price. Conversely, splitting the order into a series of smaller trades will result in smaller price pressure but more uncertainty about the execution price. As in Engle and Ferstenberg (2007), the above model integrates the portfolio decision and execution decision into a single problem.

Another strength of the Gărleanu and Pedersen (2013) model is that it accommodates both slow- and fast-moving signals in one model. The predictive signals in $f_t$ can be both slow-moving signals such as a value factor, and faster moving signals such as a short-term momentum or reversal factor. These different predictors have different mean-reversion speeds—also known as alpha-decays—which the investor must take into account when balancing the potential return against trading costs. The alpha decay determines how long the investor can enjoy high expected returns, and the investor therefore trades more aggressively on persistent signals than on signals with a high alpha decay. This tradeoff between signals with high and low alpha decay can only be analyzed in a dynamic setting.

The presence of transaction costs is important; in the absence of a cost of trading the investor can optimally rebalance his portfolio to the myopic Markowitz portfolio in each period, and the multi-period problem reduces to a series of single-period problems. With transaction costs it is costly for the investor to change his portfolio, and he therefore tries to predict his optimal future portfolio, shifting his current holdings towards the expected optimal portfolio tomorrow.

Gărleanu and Pedersen (2013) solve for the optimal portfolio holdings $x_t$ in closed form, showing that the optimal portfolio is a weighted average of 1) the current portfolio (to reduce trading costs), 2) the optimal portfolio in the absence of trading costs, and 3) the expected optimal portfolio in the future. In other words, the optimal portfolio is a weighted average of the current portfolio and an ‘aim portfolio’ that combines portfolios 2) and 3).

When the trading cost is proportional to the amount of risk, $\Lambda = \lambda \Sigma$, the optimal new portfolio $x_t$ is a weighted average of the current position $x_{t-1}$ and the aim portfolio,

$$x_t = \left(1 - \frac{a}{\lambda}\right)x_{t-1} + \frac{a}{\lambda}x_{\text{aim},t}, \quad (8)$$
where $\frac{a}{\lambda} < 1$ and

$$\text{aim}_t = (\gamma \Sigma)^{-1} G \left( I + \frac{a\beta}{\gamma} \Phi \right)^{-1} f_t$$  \hspace{1cm} (9)

$$a = -\frac{(\gamma + \lambda(1 - \beta)) + \sqrt{(\gamma + \lambda(1 - \beta))^2 + 4\gamma \lambda \beta}}{2\beta}$$  \hspace{1cm} (10)

The myopic Markowitz portfolio is simply $(\gamma \Sigma)^{-1} G f_t$. Thus, the aim portfolio is the Markowitz portfolio built as if the signals $f$ were scaled down based on their mean-reversion $\Phi$ (Gârleanu and Pedersen, 2013, Proposition 4).

### 3.2 Robust Control in a Dynamic Model

The model tries to approximate the real world faced by the investor, and all parts of the model are merely approximations of the true (unknown) model. I focus on modeling robustness with respect to a particular part of the model which is the dynamics of the predictors. Since the dynamics of the predictors are important in determining the aim portfolio, misspecified dynamics will result in the investor trading toward a sub-optimal aim portfolio. The robust investor seeks a trading strategy that performs well under a range of alternative dynamics for the predictors.

I still assume that the expected return $\mu_t = G f_t$ is known to the investor at time $t$ such that the investor always has an unbiased estimate of tomorrow’s return. In particular, the investor will behave is if $G$ is known for sure, such that—upon observing the current value of the predictors—the expected return for tomorrow is known for sure. The investor is concerned with the future evolution of the predictors which leads to biased estimates of future returns. That is, he is concerned about estimation error in $\Phi$ and he possibly distrusts the AR(1) assumption on the predictors. This differs from the single-period robust optimization problem considered in Garlappi et al. (2007).

For now, assume that (as above) the expected return is given by $\mu_t = G f_t$, such that the expected return from time $t$ to $t + 1$ is known at time $t$. The investor models the evolution of the predicting factors $f_t$ by (as above)

$$\mu_t = G f_t$$  \hspace{1cm} (11)

$$f_{t+1} = (I - \Phi)f_t + C \varepsilon_{t+1},$$  \hspace{1cm} (12)

where $V(\varepsilon_{t+1}) = I$. The investor is concerned with dynamic misspecifications of this model,
denoted the benchmark model.

Robust control theory introduces an ‘evil agent’ who aims to minimize the investor’s utility. The evil agent does this by changing the one-step-ahead transition distributions for $f_{t+1}$ given $f_t$, subject to a restriction on the size of the distortions. The distorted transition distributions are restricted to be absolutely continuous with respect to the benchmark distribution, such that the perturbed distribution cannot assign positive probability to events that have zero probability under the benchmark model. Hansen et al. (2006) show that when the transition dynamics are linear with Gaussian disturbances and the objective function is quadratic, the optimal distortion for the evil agent is to change the mean and variance of the transition density. Intuitively, because the objective function is quadratic the evil agent only spends energy on distorting the mean and variance of the distribution, and because the benchmark distribution is assumed to be normal, the worst-case distribution will again be normal. Further, Hansen et al. (2006) argue that the change in the variance is typically small and can be neglected. As a result, for linear quadratic Gaussian (LQG) problems, robust control theory replaces the iid. shocks $\varepsilon_{t+1}$ with $\varepsilon_{t+1} + w_{t+1}$, where $w_{t+1}$ is permitted to be a function of current and past values of the state vector. This allows for different forms of misspecified dynamics, including misspecified or time-varying parameters as well as higher-order dynamics. This form of model misspecification is known as unstructured misspecification.

The benchmark model is the investor’s best estimate of the true model, and he thus restricts the alternative models to be close the the benchmark model in the sense that the discounted value of all future distortions is bounded.

$$E_0 \sum_{t=0}^{\infty} \beta^{t+1} w'_{t+1} w_{t+1} \leq \eta_0.$$  \hspace{1cm} (13)

As Hansen et al. (2006) show, this is precisely a restriction on the relative entropy of the distorted distribution relative to the benchmark distribution.

The bound on the size of the distortions gives rise to a ‘model confidence region’ of alternative models. Some models within the confidence set will make the investor better off, some will make him worse off. An example of a model that makes the investor better off is a model in which the predictive signals are more persistent than the investor thought. When the investor observes an innovation to a signal, predicting a higher expected return for one or more assets, he increases his positions in these assets. However, he expects the signal, and hence the expected return, to mean-revert to zero. If the signal is more persistent than the
investor thought, the expected return will be high longer than anticipated, which is a positive surprise for the investor. Of course, had the investor known the signal was more persistent he would have traded more aggressively into the assets upon observing the signal. Likewise, a model makes the investor worse off if the signal mean-reverts more quickly to zero than expected, in which case the expected return will disappear faster than the investor thought it would. The investor is now exposed to risk from his non-zero position and has incurred trading costs by increasing his positions in the assets, but he profits less than expected from a higher return on these assets.

Robust control theory chooses a decision rule that is the optimal response to the worst possible model within the model confidence set. To obtain this, the evil agent enters the optimization problem with the objective of minimizing the investor’s utility. This leads to the following two-player zero-sum game

\[
\begin{align*}
\max_{(x_t)_{t=0}^\infty} \min_{(w_{t+1})_{t=0}^\infty} & \quad E_0 \sum_{t=0}^\infty \beta^t \left( x_t' \mu_t - \frac{\gamma}{2} x_t' \Sigma x_t - \frac{1}{2} \Delta x_t' \Lambda \Delta x_t \right) \\
\text{s.t.} & \quad \mu_t = Gf_t \\
& \quad f_{t+1} = (I - \Phi)f_t + C(\varepsilon_{t+1} + w_{t+1}) \\
& \quad E_0 \sum_{t=0}^\infty \beta^{t+1} w_{t+1}' w_{t+1} \leq \eta_0.
\end{align*}
\]

The evil agent chooses the distortions \( w_{t+1} \) to the dynamics of \( f_t \), but is restricted by a limit on the discounted size of all future distortions. This joint constraint on all future values of \( w_{t+1} \) makes the problem hard to solve as written. Fortunately, Hansen et al. (2006) show that the decision rules for the investor and the evil nature are the same in the following so-called multiplier game\(^4\)

\[
\begin{align*}
\max_{(x_t)_{t=0}^\infty} \min_{(w_{t+1})_{t=0}^\infty} & \quad E_0 \sum_{t=0}^\infty \beta^t \left( x_t' \mu_t - \frac{\gamma}{2} x_t' \Sigma x_t - \frac{1}{2} \Delta x_t' \Lambda \Delta x_t + \beta \theta w_{t+1}' w_{t+1} \right) \\
\text{s.t.} & \quad \mu_t = Gf_t \\
& \quad f_{t+1} = (I - \Phi)f_t + C(\varepsilon_{t+1} + w_{t+1}).
\end{align*}
\]

This formulation also has an intuitive interpretation: The evil agent aims to minimize the investor’s utility, but is penalized by the term \( \theta w_{t+1}' w_{t+1} \) in the objective function; every

\(^4\)Note that while the multiplier game leads to the same decision rules, the investor’s value function must be calculated based on the original objective function.
time the evil agent distorts the investor’s model, the investor is compensated with a ‘bonus’ \( \theta w_{t+1}' w_{t+1} \). A large value of \( \theta \) makes it difficult for the evil agent to lower the investor’s utility, and by making \( \theta \) arbitrarily large we approximate the optimal non-robust solution to the original problem. For each value of \( \eta \) in the problem (14)-(17) there is a corresponding value of \( \theta \) in the multiplier game (18)-(20) that results in identical decision rules for the two games. However, this mapping will not be important, and from now on I will work with the multiplier game in which the desire for robustness is parameterized by \( \theta \).

In standard dynamic programming, the certainty equivalence principle ensures that we can ignore the volatility matrix \( C \) when calculating the optimal decision rule. This is no longer true for the robust dynamic programming problem, because the evil agent’s decisions will be amplified by \( C \). However, we can still find the optimal decision rule by solving the corresponding deterministic problem, that is, by setting \( \varepsilon_t \equiv 0 \). This is known as the modified certainty equivalence principle. The problem is then

\[
\max_{(x_t)_{t=0}^\infty, (w_{t+1})_{t=0}^\infty} \min_{\beta} \sum_{t=0}^\infty \beta^t \left( x_t' \mu_t - \frac{\gamma}{2} x_t' \Sigma x_t - \frac{1}{2} \Delta x_t' \Lambda \Delta x_t + \beta \theta w_{t+1}' w_{t+1} \right)
\]

\[\mu_t = G f_t \tag{22}\]

\[f_{t+1} = (I - \Phi) f_t + C w_{t+1}. \tag{23}\]

The state variables are \( x_{t-1} \), and \( f_t \), and the control variables are \( x_t \) and \( w_{t+1} \). \( f_t \) is a state-variable because it predicts returns over the next period, and \( x_{t-1} \) is a state-variable because it is costly to change the portfolio from its composition last period.

Hansen et al. (2006) discuss how robust control theory puts a lower bound on the value function. For any dynamic model and any given trading strategy, the value function summarises the discounted utility of the trading strategy to the investor. Suppose one investor follows the optimal non-robust trading strategy and has the value function \( V^O \), and that another investor follows the robust strategy for a given level of robustness and has the value function \( V^R \). If the benchmark model is in fact the correct model, the investor following the optimal strategy will be better off, and \( V^0 > V^R \). However, if the benchmark model is not the correct model, the value functions change. As the amount of model misspecification increases, the value functions decrease. The robust trading strategy is such that the robust value function decreases more slowly than the non-robust value function, thus guaranteeing a lower bound on the value function for a given amount of model misspecification. Eventually, as the distortions grow larger, \( V^R > V^O \), and the robust investor is better off. Robust
control is a form of insurance. If the investor knows the true data generating process he will lose money by following the robust strategy. On the other hand, he gains when the data generating process deviates from his benchmark model.

4 Solution

The robust solution can be found by solving a standard linear quadratic Gaussian dynamic programming problem as demonstrated in Appendix A.1. The value function takes the form

$$V(x_{t-1}, f_t) = -\frac{1}{2} x'_{t-1} A_{xx} x_{t-1} + x'_{t-1} A_{xf} f_t + \frac{1}{2} f'_{t} A_{ff} f_t + a_0$$  \hspace{1cm} (24)

and the optimal decision rules for the investor and the evil agent can be written as

$$x_t = -F(x_{t-1})$$  \hspace{1cm} (25)

$$w_{t+1} = K (x_{t-1}) = K_{x} x_{t-1} + K_{f} f_t$$  \hspace{1cm} (26)

The solution is easy to compute numerically and only requires solving a Ricatti equation. I characterize the robust portfolio choice in the following sections. Section 4.1 describes the decision rule of the evil agent, Section 4.2 and 4.3 describe the robust trading strategy, and Section 4.4 illustrates how the robust trading rule depends on the parameters of the model. Proofs are in the appendix.

4.1 The evil agent

To gain intuition for the robust portfolio choice, I first consider the behaviour of the evil agent. The evil agent follows the decision rule $w_{t+1} = K_{x} x_{t-1} + K_{f} f_t$. As a result, under the worst-case model, the predictors follow the dynamics

$$f_{t+1} = (I - \Phi) f_t + C(\varepsilon_{t+1} + w_{t+1}) = (I - \Phi + C K_{f}) f_t + C K_{x} x_{t-1} + C \varepsilon_{t+1}. \hspace{1cm} (27)$$

Note that $f$ no longer follows a simple AR(1) process as the dynamics now depend on $x_{t-1}$. There are two changes to the dynamics of $f$ under the worst-case model. First, the mean-reversion coefficient is changed from $\Phi$ to $\Phi - C K_{f}$. This has the effect of making the predictors mean-revert faster such that the expected excess returns disappear faster. This is
a systematic effect, and as a result the investor behaves as if he has a more negative view on
future returns. Second, the distortion depends on the current position $x_{t-1}$. The evil agent
can only distort the model within a given limit (it can spend a certain amount of entropy),
and it must choose wisely how to hurt the investor the most within this limit. It therefore
chooses to distort the model the most when the investor has taken a large position, which is
when the investor is most vulnerable to misspecification. From the investor’s point of view,
this captures the increased concerns an investor faces when trading into a large position. If
the investor starts with an initial position of zero in an asset, he is not too worried about
misspecified dynamics of the predictor—if the signal changes he is unaffected since he has a
zero position. But as he takes on an increasingly large position, he becomes more worried
about misspecification. If the excess return were to disappear, he would be left with a large
risk and no excess return, and would incur large trading costs to trade out of the position
again. The worst-case dynamics capture the investor’s increased concern for misspecification
as the position is increased.

4.2 Super-Robustness
To characterize the robust solution, I first consider the case in which the investor has an
infinite desire for robustness. I call this super-robustness. In this case, the investor knows
the expected return for tomorrow, which is $G_{f_t}$, but he expects the evil agent to distort the
signal in the worst possible way the day after tomorrow. Since the evil agent distorts the
dynamics of the predictors it only has the ability to change the expected return after the
next period. As a result, the investor plans to close out all positions after tomorrow, such
that he holds a zero position in all assets when the evil agent distorts the predictor. The
problem now becomes a two period problem. The investor wants to trade into the position
today, capture the known expected return tomorrow, and he then plans to trade out of the
position before the evil agent distorts the model.

**Proposition 4.1.** When the investor has an infinite desire for robustness, the optimal po-
sition is given by

$$x_t = (\gamma \Sigma - 2\Lambda)^{-1}(\gamma \Sigma \times \text{Markowitz}_t + \Lambda x_{t-1}).$$  \hspace{1cm} (28)

The super-robust solution does not depend on the mean-reversion of the factors. The
intuition is straightforward. The optimal position is a combination of the previous position
$x_{t-1}$ and the Markowitz position $(\gamma \Sigma)^{-1}G_{f_t}$. On the first day, the investor trades from his
previous position $x_{t-1}$ to this super-robust position and captures the expected return. On
the second day, the investor plans to close the position. The change in position each day is
determined by the size of the expected return $Gf_t$ and the transaction costs $\Lambda$. In particular,
if transaction costs are zero the investor simply trades to the Markowitz portfolio which has
the optimal risk-return tradeoff, and then closes out the position the next day. On the other
hand, when transaction costs are extremely high the investor trades halfway from his current
position $x_{t-1}$ towards zero on the first day, and then all the way to zero on the second day.

[Figure 1 about here.]

In the general case with robustness, the investor moves from the optimal non-robust
position toward the super-robust position as the desire for robustness is increased. Figure 1
illustrates this. There are two assets, and the expected return on each asset is described
by its own predictor. The two signals are equally strong, and hence the current Markowitz
portfolio has equal positions in the two assets. The predictor for asset 1 is slowly mean-
reverting, whereas the predictor for asset 2 is quickly mean-reverting. Hence, asset 1 poses a
more attractive investment than asset 2 since the positive return is expected to last longer.
The optimal position thus has a larger position in asset 1 as a result. As the desire for
robustness is added, the robust position moves from the optimal position toward the super-
robust position.

As the desire for robustness is increased, the robust portfolio does not move directly from
the optimal portfolio toward the super robust portfolio. The reason is that, when the evil
agent is limited in the size of the distortion, the evil agent hurts the investor the most by
distorting the predictor for asset 1 (the attractive investment) more than the predictor for
asset 2, such that the robust position initially moves toward the diagonal. In Section 4.4 I
discuss in more detail how the robust solution depends on the parameters of the model.

### 4.3 Aim in Front of the Target

In the model without robustness, Gârleanu and Pedersen (2013) show that the investor
trades partially toward an aim portfolio which is a weighted average of the current Markowitz
portfolio and the expected Markowitz portfolios at all future times. The optimal position is
(Gârleanu and Pedersen, 2013, Proposition 2)

$$ x_t = x_{t-1} + \Lambda^{-1} A_{xx} (\text{aim}_t - x_{t-1}) \tag{29} $$

where the aim portfolio is given by $\text{aim}_t = A_{xx}^{-1} A_{xf} f_t$. Alternatively, the aim portfolio can
be written as a weighted average of the current and expected future Markowitz portfolios as
in Gârleanu and Pedersen (2013, Proposition 3):

$$\text{aim}_t = \sum_{\tau=t}^{\infty} z(1-z)^{\tau-t} E_t(Markowitz_{\tau})$$  

(30)

A similar characterization is possible for the robust solution. Here, the investor again trades partially toward an aim portfolio and the aim portfolio is related to the expected future Markowitz portfolios. There are two modifications for the robust solution. First, the expectation is taken under the worst-case dynamics for which the mean-reversion of the predictors is $\Phi - CK_f$. Second, since the distortion also depends on the investor’s current position $x_{t-1}$, the aim portfolio similarly must depend on $x_{t-1}$.

To gain intuition, recall that the evil agent distorts the model according to $w_{t+1} = K_x x_{t-1} + K_f f_t$. Suppose for a moment that the distortion only depends on $f_t$ such that $w_{t+1} = K_f f_t$. In this case, the mean-reversion of the predictors are changed and the new dynamics of the predictors are $f_{t+1} = (I-\Phi+CK_f)f_t$. The results in Gârleanu and Pedersen (2013) provide us with the optimal strategy under this model: trade partially toward the aim, which is a weighted average of the current and expected future Markowitz portfolios under the new dynamics. This intuition captures the main effect of robustness. However, since the distortion also depends on the investor’s position, this must be incorporated into the robust trading strategy as well. Simply aiming at a weighted average of the current and expected future Markowitz portfolios under the new dynamics does not take into account that the choice of position feeds back into the dynamics of the predictors. Therefore, the robust trader adjusts his targets and aim to account for this feedback. Proposition 4.2 shows that the robust investor trades partially toward the robust aim portfolio, and Proposition 4.3 characterizes the robust aim portfolio.

**Proposition 4.2. (Trade Partially Toward the Aim)**

The robust portfolio is

$$x_t = x_{t-1} + (\Lambda - \beta K_x' C' A'_{xf})^{-1} A_{xx}(\text{aim}^R_t - x_{t-1})$$  

(31)

which implies trading at a proportional rate given by the matrix $(\Lambda - \beta K_x' C' A'_{xf})^{-1} A_{xx}$ toward the robust aim portfolio, where $\text{aim}^R_t = \text{aim}^R_f f_t + \text{aim}^R_x x_{t-1}$ and

$$\text{aim}^R_f = A_{xx}^{-1} (A_{xf} + 2\beta \theta K_x' K_f + \beta K_x' C' A_{ff}(I-\Phi+CK_f))$$  

(32)

$$\text{aim}^R_x = A_{xx}^{-1} (\beta K_x' C' A'_{xf} - 2\beta \theta K_x' K_x - \beta K_x' C' A_{ff} CK_x)$$  

(33)
Compared to the optimal solution in which the trading speed depends only on $\Lambda$, the effect of robustness is to trade less toward the aim. More importantly, the aim portfolio is also different in the case with robustness. In the non-robust case, the aim portfolio is given by $A_{xx}^{-1}A_{xf}f_t$. Here, in the robust case, the loading on $f_i$ has changed because the predictors follow different mean reversion dynamics under the worst case model. Finally, the current aim also depends on the current position $x_{t-1}$. Since the evil agent distorts the model depending on the investor’s position, the investor is more reluctant to take large positions.

The part of the aim portfolio that depends on the predictors, $aim^R_f f_t$, can also be characterized as a weighted average of the current Markowitz portfolio and expected future targets. In the model without robustness, the future targets are simply the future expected Markowitz portfolios. With robustness, the expectation is now taken under the worst-case dynamics for which the mean-reversion of the predictors is $\Phi - CK_f$. Also, with robustness the investor knows that his choice of position will feed back into the dynamics of the predictors, so he must adjust his target accordingly. As a result, $aim^R_f f_t$ is an average of the future expected Markowitz portfolios, plus an offset which is proportional to the expected predictors.

**Proposition 4.3. (Aim in Front of the Target)**

The part of the robust aim portfolio that is a function of $f_t$, $aim^R_f f_t$, is the weighted average of the current and future target portfolios, plus an offset,

$$aim^R_f f_t = A_{xx}^{-1}2N_{sf}f_t + A_{xx}^{-1}\gamma \Sigma \sum_{s=0}^{\infty} \beta^s Z^s \times Target^R_{t+s}.$$  

The current robust target portfolio is the current Markowitz portfolio, and the future robust target portfolios are given by the expected future worst case Markowitz portfolios plus an offset that is proportional to the expected value of the predictors,

$$Target^R_{t+s} = E^w(Markowitz_{t+s}) + (\gamma Z)^{-1}N_{sf}E^w(f_{t+s})$$

Here, $E^w$ denotes the expectation under the worst-case model in which the mean reversion of the predictors is $\Phi - CK_f$.

**4.4 Examples**

The evil agent’s distortion of the signal is linear in both the size of the position and the magnitude of the signal, but the coefficients depend on the parameters of the model. Since the evil agent can only distort the model within certain limits, he distorts the signal more
when \( i \) the signal is more persistent and \( ii \) the variance of the innovations to the signal is larger.

The first effect was already discussed in Section 4.2 above. Since a signal from a highly persistent predictor is more valuable to the investor than a signal from a less persistent predictor, the evil agent hurts the investor the most by distorting the signals from the most persistent predictors. The effect of the persistence of the signal is illustrated in Figure 2. There are two assets, and the expected return on each is described by its own predictor. The predictor is slowly mean-reverting for asset 1 and quickly mean-reverting for asset 2. The myopic Markowitz positions, starting in the top right of the figure, converge to zero over time, but converge more quickly for the second asset. Therefore, the investor wants to take a larger position in asset 1, and without robustness the investor trades toward the optimal aim portfolio to \( x_{t}^{\text{opt}} \). With robustness, the target portfolios are changed to reflect the mean-reverting dynamics under the worst-case model. Since the evil agent distorts the signal for the attractive asset 1 the most, the new target portfolios lie closer to the diagonal and move more quickly toward zero. The robust aim portfolio is given in Proposition 4.2 and 4.3, and is a weighted average of the new target portfolios plus an offset depending on the current position \( x_{t-1} \).

[Figure 2 about here.]

For the second point, recall that the distortion \( w_{t+1} \) is scaled by the volatility matrix \( C \), in \( f_{t+1} = (I - \Phi)f_{t} + C(\varepsilon_{t+1} + w_{t+1}) \). It is thus more beneficial for the evil agent to distort signals that have a high variance of the innovations. Consequently, the variance of the innovations matter for the robust solution. In contrast, for the non-robust optimal solution the certainty equivalence principle ensures that the solution does not depend on the volatility matrix \( C \). Intuitively, there is more uncertainty about the future evolution of signals with high volatility of the innovations and the robust investor trades more cautiously on these signals.

The effect of the variance to the innovations of the signal is illustrated in Figure 3. As before, there are two assets, and the expected return on each is described by its own predictor. The predictors now have the same mean reversion speed, but the innovations to the first predictor have a higher variance than the innovations to the second predictor (the innovations are uncorrelated). The myopic Markowitz positions, starting in the top right of the figure, converge to zero over time and the expected Markowitz portfolios have equal positions in the two assets. Indeed, the optimal solution does not depend on the variance of
the predictors (the certainty equivalence principle). The robust solution, however, depends on the variance of the innovations to the predictors. Since the innovations to predictor 1 have a higher variance, it is optimal for the evil agent to distort this predictor more. As a result, the new target portfolios have smaller positions in asset 1 than in asset 2, and the robust aim portfolio consequently also takes a smaller position in asset 1.

[Figure 3 about here.]

5 Scaled Optimal Strategies

As described above, the effect of robust control is that the investor trades less toward the aim and that the aim is changed. The change in the aim reflects the mean reversion dynamics under the worst-case model as well as the current position of the investor. As the investor moves from the optimal non-robust position to the super-robust position, the volatility of the strategy decreases. In this section I demonstrate that by scaling the optimal (non-robust) trading strategy, its volatility can be lowered as well. However, the resulting trading rules are not identical to the robust trading rule, and the empirical results in Section 6 demonstrate that when scaled to the same volatility as the robust strategy, the scaled optimal strategies perform worse out-of-sample when transaction costs are large.

There are three ways in which the optimal non-robust strategy can be scaled to a lower volatility: (1) by scaling the expected returns, (2) by scaling the risk aversion of the investor, and (3) by scaling the mean-reversion of the predictors. I describe each method here and analyze their out-of-sample performance in Section 6.

Recall that the optimal non-robust strategy is given by

\[ x_t = \left(1 - \frac{a}{\lambda}\right)x_{t-1} + \frac{a}{\lambda}\text{aim}_t, \tag{36} \]

where \( \frac{a}{\lambda} < 1 \) and

\[ \text{aim}_t = (\gamma \Sigma)^{-1} G \left( I + \frac{a\beta}{\gamma} \Phi \right)^{-1} f_t. \tag{37} \]

The simplest way of lowering the volatility of the optimal non-robust strategy is to shrink the expected return, i.e., to scale \( G \) in \( \mu_t = G f_t \). This scales down the current and future expected myopic Markowitz portfolios. As a consequence, multiplying \( G \) by, say, 0.5 gives a sequence of portfolio allocations that are exactly half of their previous value.\(^5\) Scaling \( G \)

\(^5\)More formally, if the two portfolios, using \( G \) and 0.5\( G \) respectively, both start with zero initial positions,
is the most naive way of controlling the volatility of the optimal non-robust strategy, and clearly any level of volatility can be targeted by scaling $G$.

A second way of lowering the volatility of the optimal non-robust strategy is to increase the investor’s absolute risk aversion. There are three effects of this. First, of course, the myopic Markowitz portfolio is scaled down. The more interesting result is how the investor tracks the Markowitz portfolio which is determined by the two other effects. Second, the aim portfolio changes, beyond the direct effect of scaling down the Markowitz portfolio. Since $\frac{a\beta}{\gamma} \rightarrow 0$ as $\gamma$ is increased, the aim portfolio moves closer to the Markowitz portfolio. Intuitively, the Markowitz portfolio is ‘safe’ in the dynamic model since it is the current optimal position. In other words, when the investor is more risk averse, he acts as if the signals are more persistent (this is what makes the aim portfolio move toward the current Markowitz portfolio). Third, since $a$ is a function of $\gamma$, $a/\lambda \rightarrow 1$ as $\gamma$ is increased. As a result, the investor trades further toward the aim portfolio. The total effect of a higher risk aversion is thus to scale down the Markowitz portfolio and to track the Markowitz portfolio more closely. Vice versa, when risk aversion is low, the Markowitz portfolio is scaled up, but we trade less toward it. Any level of volatility can be targeted by adjusting $\gamma$.

A third way of lowering the volatility of the optimal non-robust strategy is to change the mean-reversion of the predictors directly. By scaling the mean-reversion matrix $\Phi$, the investor behaves as if the predictors mean-revert faster. As a result, he changes his aim toward zero and takes smaller positions. The most pessimistic outlook, within the AR(1) framework of the benchmark model, would be if the predictors are perfectly oscillating, i.e., if $\Phi = 2I$ such that $f_{t+1} = -f_t + C \varepsilon_{t+1}$. In this case, any positive expected return tomorrow is followed by a negative expected return of equal magnitude on the following day. The robust strategy changes the mean-reversion of the predictors more for more persistent predictors. To mimic this, consider a scaling in which the mean-reversion matrix is set to $\Phi' = \Phi - \delta(\Phi - 2I), \delta \in (0, 1)$. Note that there is no guarantee that a specific level of volatility can be obtained, since it is still optimal for the investor to capture the known expected return tomorrow and then plan to close out his position at the end of the day.

None of above ways of scaling the original problem leads to the robust solution. Consider the examples in Section 4.4. When the two predictors have different mean-reversions, the effect of robustness is to scale down the more persistent signal more quickly, such that the robust aim portfolio moves toward the diagonal as well as the origin as in Figure 2. The first portfolio will always have a position of one-half of the second. If the two portfolios have non-zero initial positions, the first will converge to one-half of the second.
effect of scaling $G$ is to move the aim portfolio linearly toward the origin. When scaling risk-aversion and predictor mean-reversion in the benchmark model, the resulting aim portfolios both move toward the diagonal but much more slowly than for the robust aim portfolio.

Moreover, the volatility of the predictors, which is important in the robust solution, does not enter into the optimal non-robust solution. Consider the second example in Section 4.4 in which the two predictors have different volatilities (see Figure 3). In this case, when scaling $G$, $\gamma$ or $\Phi$ in the benchmark model, the optimal aim portfolio moves directly toward the origin. None of these solutions take into account that the two predictors have different volatilities. In contrast, the robust aim portfolio moves away from the diagonal since there is more uncertainty about the future of the first predictor.

6 Empirical Application: Robust Momentum Strategies

6.1 Data Sets

In this section I backtest a robust momentum strategy on historical data. Since the robust strategy requires estimating the covariance matrix $\Sigma$, I focus on asset classes with relatively few assets. The backtest is based on daily excess returns. I obtain the daily risk-free return from Ken French’ web site. I test the strategy on four different data sets and rebalance all portfolios daily. All tests are out-of-sample. I use the same parameters (described below) for all asset classes, and the exact same code is executed to construct the robust momentum portfolio for each asset class—the only thing that changes is the return series.

6.1.1 10 Industry Portfolios

I obtain daily data on the 10 industry portfolios from Ken French’ web site and subtract the risk-free rate to calculate excess returns. The sample is daily returns over the period 1980:01-2013:12.

6.1.2 International Indices

I obtain data on the following 18 international indices from DataStream; Australia, Austria, Belgium, Canada, Denmark, France, Germany, Hong Kong, Italy, Japan, Netherlands, Norway, Portugal, Spain, Sweden, Switzerland, UK, and the US. DataStream provides a
‘price index’ which excludes dividends, as well as a ‘return index’ including dividends. The
return index is monthly until 2001, after which it becomes daily. To construct daily returns
with dividends prior to 2001, I adjust the daily return from the ‘price index’ such that the
cumulative monthly returns match the monthly returns of the ‘return index.’ All returns
are in USD, and I subtract the daily risk-free rate to calculate excess returns. The sample
is daily returns over the period 1980:01-2013:12. The data for 17 of the 18 countries are
available for the full sample, and data for Portugal start in 1988:01.

6.1.3 Commodities

I obtain data on the following 24 commodity futures from Bloomberg: Crude Oil, Gasoil,
WTI Crude, Unl. Gasoline, Heating Oil, Natural Gas, Cotton, Coffee, Cocoa, Sugar, Soy-
beans, Kansas Wheat, Corn, Wheat, Lean Hogs, Feeder Cattle, Live Cattle, Gold, Silver,
Aluminum, Nickel, Lead, Zinc, Copper. I use the Goldman Sachs commodity indices from
Bloomberg, which account for the roll. Also, the returns are excess returns which effectively
assumes that the positions are fully collateralized and ignores interest received on the col-
lateral. The sample is daily returns over the period 1980:01-2013:12. 10 commodities are
available over the full sample, and all are available from 2002:01.

6.1.4 Foreign Exchange

I obtain daily data on the exchange rate for the following 9 countries: Australia, Canada,
Euro, Japan, New Zealand, Norway, Sweden, Switzerland, and the UK. The exchange rate
for the Euro has been spliced with the D-Mark. Currency strategies are usually studied
using monthly spot and forward rates. However, I require daily returns, so I obtain the 1
month interest rate for the US as well as the above countries. The exchange rates are ‘foreign
currency/USD.’ To construct daily returns on investing in foreign currency, I calculate the
equivalent daily interest rates and compute the return for country $n$ as

$$ r_{t+1}^n = (1 + i_t^n) \frac{S_t^n}{S_{t+1}^n} - (1 + i_t^US), \tag{38} $$

where $S_t^n$ is the exchange rate on date $t$, $i_t^n$ is the daily foreign interest rate, and $i_t^US$ is
the daily US interest rate. The sample is daily returns over the period 1986:01-2013:12. 8
countries are available over the full sample, New Zealand is added in 1988:03, and Sweden
is added in 1992:12 (due to limited availability of interest rates).
6.1.5 All Assets Classes

I first form a robust momentum portfolio for each of the 4 asset classes above. In addition, I consider a portfolio that combines the four individual momentum portfolios. I construct this by scaling each individual momentum portfolio to have a volatility of 10% and then forming an equal weighted portfolio of these scaled portfolios. This portfolio starts in 1980:01, and currencies are then added in 1986:01.

6.2 Estimating the Model

The signal for an asset is based on the asset’s log return over the previous year, skipping the most recent month. This is standard in the momentum literature. In the dynamic trading problem, I shrink the expected return halfway towards zero, such that the $G$ matrix is 0.5 times the identity matrix (a constant picks up the unconditional historical average, so only the momentum effect is shrunk).

The mean-reversion matrix $\Phi$ is estimated as a diagonal matrix, where each entry is found by regressing the the predictor onto its lag. A constant predictor is added to account for non-zero average returns.

All variances and covariances are estimated using an exponential moving average, in which half of the weight is on the most recent 60 days. In particular, $\Omega$ (with Cholesky decomposition $C$) and $\Sigma$ are both estimated using exponential averages.

I set $\gamma = 10^{-9}$, corresponding to a relative risk aversion of 1 for an investor with 1BN under management. I do not change the absolute risk aversion as the investor gains or loses money, but keep it fixed at $\gamma = 10^{-9}$. An alternative way of implementing the model is to think of the relative risk aversion as fixed, and then adjust the absolute risk aversion as the NAV changes. The drawback of this approach is that as the investor becomes richer he wants to trade more, but the quadratic transaction costs makes this very expensive. Hence, as a fraction of wealth, the investor trades less and less as he becomes richer, and this makes it difficult to evaluate the performance over long periods of time. Instead, I fix $\gamma$ such that the investor always behaves the same way. When evaluating the performance below, I calculate daily dollar returns (after transaction costs), and convert them to percentage returns by scaling by 1BN.

I model the transaction cost matrix $\Lambda$ as $\Lambda = \lambda \Sigma$, which also accounts for cross-asset price impact. I consider three values of $\lambda$. In a medium trading cost environment I set $\lambda = 5 \times 10^{-7}$ as in Gärleanu and Pedersen (2013). In addition, I consider a high- and low
cost setting in which $\lambda = 5 \times 10^{-5}$ and $\lambda = 5 \times 10^{-9}$, respectively, such that trading is either 100 times cheaper or 100 times more expensive. This can be interpreted either as higher transaction costs for a given trade size, or as the cost of trading for a smaller or larger investor. As $\Sigma$ is re-estimated over time, $\Lambda$ changes. In particular, it is more costly to execute a given trade in volatile times than in more quiet times.\footnote{The dynamic programming problem assumes that $\Lambda$ and $\Sigma$ are constant over time. Re-estimating them, however, likely corresponds to how an active investor would implement the model. It is not possible to model $\Sigma$ as time-varying within a linear-quadratic model.}

I use the same values for all asset classes such that I run the exact same code for all out-of-sample tests.

### 6.2.1 Choosing the Desire for Robustness

An investor seeking robustness must choose the desired degree of robustness. Too little, and the investor simply follows the optimal non-robust strategy. Too much, and the investor constantly takes the super-robust position without regard for the mean-reversion of the signals. The desire for robustness can potentially be calibrated based on model confidence regions, such that the investor seeks robustness with respect to models that are statistically hard to distinguish from the estimated model. I take a more pragmatic approach. I assume that the investor simply wants a strategy with a 10\% annualized volatility. Without a desire for robustness, the optimal level of volatility would be determined jointly with the optimal expected return by the investor’s indifference curves. Robustness introduces another dimension to the investor’s preferences. His risk aversion is unchanged, but I assume that because he distrusts the model he is not willing to engage in a momentum strategy with more than 10\% annualized volatility. He then simply chooses the degree of robustness such that the conditional volatility of the strategy (which can be easily calculated from the positions and the covariance matrix of returns) is 10\%, and he adjusts his desire for robustness over time to keep the volatility at 10\%.

This is a simple and transparent rule for selecting the desire for robustness. With a low desire for robustness, the investor approaches the optimal non-robust strategy which has a very high volatility, and with a very high desire for robustness, the investor approaches the super-robust position which has a very low volatility.
6.3 Myopic Markowitz Positions, Scaled Optimal Strategies, and Rank-Weighted Strategies

I compare the out-of-sample performance of the robust strategies to several alternative strategies. First, I consider a strategy where the investor each day takes the myopic Markowitz position and does not incur trading costs. The Markowitz positions are scaled each day to an annualized volatility of 10%. This corresponds to an ideal world without transaction costs, and the performance of the Markowitz strategy is naturally superior to all other strategies. This also demonstrates that there is out-of-sample predictability in all asset classes I consider.

Next, I compare the robust strategy to the scaled optimal strategies described in Section 5. In each case, the parameters of the benchmark model—\( \gamma \), \( \Phi \), and \( G \), respectively—are scaled such that the annualized volatility of the resulting portfolio is 10%.

Finally, I also form a rank-weighted momentum portfolio where the position in each asset is determined by the relative magnitude of the asset’s past return: Letting \( f_i \) be the signal for asset \( i \), the weight in asset \( i \) is then \( \text{rank}(f_i) - \frac{1}{N} \sum_1^N \text{rank}(f_i) \), that is, the asset’s rank minus the average rank. In particular, the strategy goes long past winners and short past losers. The rank based portfolios are rebalanced monthly, as is standard in that literature. I report the performance of the rank-based strategy without subtracting transaction costs. To consider the performance of the rank based strategy net of transaction costs, I would have to take an explicit stand on how to rebalance the portfolio, and I prefer not to do this.

6.4 Out-of-Sample Performance

I report the performance of the dynamic strategies after transaction costs. For all strategies, I first compound the daily returns for each portfolio into monthly returns and base all performance metrics on these monthly returns. For each asset class I construct an equal weighted index, and I calculate alphas and betas relative to this index.

Table 1 shows the Sharpe ratio, \( \alpha \), and \( \beta \) of the different strategies. Panel A shows that the myopic Markowitz strategy performs very well in terms of Sharpe ratio and alpha. Across all four asset classes, the myopic Markowitz strategy generates an alpha that is significant at the 1% level. Clearly, momentum has strong out-of-sample predictability for all asset classes, including currencies. For comparison, the rank-weighted momentum strategy also generates a significant alpha for three out of the four asset classes, but the alphas are smaller than for the myopic Markowitz portfolio. Combining all asset classes, the myopic Markowitz
strategy generates a Sharpe ratio of 1.18 and an alpha of 11.4 p.a., compared to a Sharpe ratio of 0.56 and an alpha of 5.5% for the rank-weighted strategy. The difference shows that incorporating the covariance structure of returns improves the performance dramatically. Of course, the myopic Markowitz portfolio suffers from tremendous turnover. The dynamic trading problem seeks to balance tracking the myopic Markowitz portfolio closely against minimizing transaction costs.

Panel B shows the performance of the robust strategy for the three levels of transaction costs. In all cases, the level of robustness is chosen such that the annualized volatility of the strategy is 10%. As the level of transaction costs decreases, the performance of the strategy improves and converges to the performance of the myopic Markowitz portfolio. The robust strategy performs well even for high levels of transaction costs. For the highest setting, the robust strategy generates alphas that are significant at the 10% level for all asset classes and at the 5% level for two out of the four asset classes.

Panel C shows the performance of the optimal strategy for which the expected return is scaled to target a 10% volatility (that is, \( G \) is scaled). For medium and high levels of transaction costs, the performance is similar to that of the robust strategy. However, when transaction costs are large, the robust strategy performs better, increasing the Sharpe ratio from 0.57 to 0.72 and increasing the alpha for the combined asset class portfolio from 4.09% to 6.10%. Of course, it is exactly when transaction costs are high that misspecification will hurt the investor greatly, and it is in this scenario that the robust portfolio offers the greatest advantage.

Scaling risk aversion or the predictor mean-reversion for the optimal strategy leads to similar results. In Panel D, I scale the risk aversion of the investor to target a 10% volatility and this improves the performance slightly compared to scaling \( G \). For large transaction costs, the robust strategy generates a higher alpha and Sharpe ratio for all asset classes. The results in Panel E shows a similar performance when scaling the mean-reversion of the predictors in the optimal strategy.

[Table 1 about here.]

In summary, the robust strategy provides insurance against misspecification of the dynamics of the predictors. These dynamics are important when trading is costly, and in this case the robust strategy outperforms the scaled optimal strategies for all asset classes.

Figure 4 plots the cumulative log returns for each of the four asset classes. Each plot shows the performance of the myopic Markowitz strategy that ignores transaction costs to the
robust strategies net of transaction costs. Naturally, the performance is generally stronger when transaction costs are low. However, this is not always the case. If the model generates an incorrect signal, causing the investor to lose money, the investor facing high transaction costs can outperform the investor facing low transaction costs since the former trades less aggressively on the signal. Similarly, the strategies with transaction costs may in some cases outperform the myopic Markowitz strategy that ignores transaction costs.

Figure 5 shows the cumulative log returns for the combined asset class strategy as well as the draw downs of the strategy. As above, the performance is better for lower levels of transaction costs, but even for the highest setting of transaction costs the robust strategy generates a high alpha of 6.1% p.a.

7 Conclusion

This paper derives the optimal dynamic trading strategy when the investor’s model of alpha-decay is misspecified. This robust trading strategy can be computed easily by solving a standard linear quadratic Gaussian dynamic programming problem. The robust strategy down-weights more persistent signals relative to less persistent signals, and down-weights signals with high innovation variance. In essence, the strategy puts less weight on signals for which misspecification will hurt the investor the most. Intuitively, the robust strategy puts a lower bound on the value function, such that it guarantees a certain (risk-adjusted) profit for a certain amount of misspecification. In contrast to the standard single-period robust optimization procedures, the robust control solution presented here takes into account dynamic misspecification of the model.

I apply the robust dynamic strategy to a standard momentum signal. Across four asset classes including industry indices, international stock indices, commodities, and currencies, the robust strategy generates a higher alpha than a standard rank-weighted momentum strategy. When combining the portfolios for the four asset classes into one, this portfolio generates a Sharpe ratio of above 1 while only trading in a total of 60 contracts.
A  Proofs

A.1  Implementation: Solving as a Standard LQG Problem

This may be written as a standard linear-quadratic regulator problem, by letting the control variable be \( u_t = (x'_t, w'_{t+1})' \) and defining

\[
A = \begin{pmatrix}
0_{n \times n} & 0_{n \times m} \\
0_{m \times n} & \Phi
\end{pmatrix}, \quad
B = \begin{pmatrix}
I_n & 0'_{n \times m} \\
0_{m \times n} & C
\end{pmatrix}
\]

\[
Q = \begin{pmatrix}
-\frac{1}{2}A & 0_{n \times m} \\
0_{m \times n} & 0_{m \times m}
\end{pmatrix}, \quad
R = \begin{pmatrix}
-\frac{\gamma}{2} \Sigma - \frac{1}{2}A & 0_{n \times m} \\
0_{m \times n} & \beta \theta I_m
\end{pmatrix}, \quad
W = \begin{pmatrix}
\frac{1}{2}A & 0_{n \times m} \\
\frac{1}{2}G' & 0_{m \times m}
\end{pmatrix}
\]

(39)

Now, the problem can be written in the usual form: Choose \((u_t)_{t=0}^\infty\) to maximize

\[
\sum_{t=0}^\infty \beta^t \left( y'_t Q y_t + u'_t R u_t + 2y'_t W u_t \right) \quad \text{s.t.} \quad y_{t+1} = A y_t + B u_t.
\]

(41)

The problem can be solved using standard techniques for linear-quadratic problems resulting in optimal decision rules for the investor and the minimizing player, given by

\[
x_t = -F y_t
\]

(43)

\[
w_{t+1} = K y_t
\]

(44)

for some matrices \(F\) and \(K\). In particular, the value function is found by solving a Riccati equation. Note that unlike Gårleanu and Pedersen (2013) I’m unable to solve for the optimal trading strategy in closed form. The reason is that the decision of the evil agent affects the future evolution of the signals. Similarly, when allowing for permanent and transitory price impact (that is, the investor’s decision feeds back into the signals) Gårleanu and Pedersen (2013) are no longer able to solve for the optimal strategy in closed form.

A.2  The Value Function

Fix the decision rule for the evil agent to be \( w_{t+1} = K y_t = K_f f_t + K_x x_{t-1} \), such that

\[
f_{t+1} = (I - \Phi) f_t + C (\epsilon_{t+1} + w_{t+1}) = (I - \Phi) f_t + C K_f f_t + C K_x x_{t-1} + C \epsilon_{t+1}.
\]

(45)
The problem is then to choose $x_t$ to maximize

$$x_t'Gf_t - \frac{\gamma}{2} x_t' \Sigma x_t - \frac{1}{2} (x_t - x_{t-1})' \Lambda (x_t - x_{t-1}) + \beta \theta w_{t+1} w_{t+1} + \beta E_t (V(x_t, f_{t+1}))$$

$$= x_t'Gf_t - \frac{\gamma}{2} x_t' \Sigma x_t - \frac{1}{2} (x_t - x_{t-1})' \Lambda (x_t - x_{t-1}) + \beta \theta (K_f f_t + K_x x_{t-1})' (K_f f_t + K_x x_{t-1})$$

$$- \frac{1}{2} \beta x_t' A_{xx} x_t + \beta x_t' A_{xf} ((I - \Phi + CK_f) f_t + CK_x x_{t-1})$$

$$+ \frac{1}{2} \beta ((I - \Phi + CK_f) f_t + CK_x x_{t-1})' A_{ff} ((I - \Phi + CK_f) f_t + CK_x x_{t-1})$$

$$+ \frac{1}{2} \beta E_t (\epsilon_{t+1}' C A_{ff} C \epsilon_{t+1})$$

$$= - \frac{1}{2} x_t' J x_t + j_t x_t + d_t$$

where

$$J = (\gamma \Sigma + \Lambda + \beta A_{xx}) \quad (46)$$

$$j_t = Gf_t + \Lambda x_{t-1} + \beta A_{xf} ((I - \Phi + CK_f) f_t + CK_x x_{t-1}) \quad (47)$$

$$= (G + \beta A_{xf} (I - \Phi + CK_f)) f_t + (\Lambda + \beta A_{xf} CK_x) x_{t-1} \quad (48)$$

$$d_t = - \frac{1}{2} x_{t-1} \Lambda x_{t-1} + \ldots \quad (49)$$

The solution is

$$x_t = J^{-1} j_t \quad (50)$$

and the value obtained is

$$- \frac{1}{2} j_t' J^{-1} J J^{-1} j_t + j_t J^{-1} j_t + d_t = \frac{1}{2} j_t' J^{-1} j_t + d_t \quad (51)$$

Matching coefficients with the value function

$$V(x_{t-1}, f_t) = - \frac{1}{2} x_{t-1}' A_{xx} x_{t-1} + x_{t-1}' A_{xf} f_t + \frac{1}{2} f_t' A_{ff} f_t + a_0 \quad (52)$$

28
A.3 Super-Robust Optimal Position

For the investor with an infinite desire for robustness, the problem is

$$\max_{x_t} (Gf_t)'x_t - \frac{\gamma}{2} x_t' \Sigma x_t - \frac{1}{2} (x_t - x_{t-1})' \Lambda (x_t - x_{t-1}) - \frac{1}{2} x_t' \Lambda x_t$$

(55)

i.e.,

$$\max_{x_t} (Gf_t)'x_t - \frac{\gamma}{2} x_t' \Sigma x_t - x_t' \Lambda x_t + x_{t-1}' \Lambda x_{t-1} - 2x_{t-1}' \Lambda x_{t-1}.$$  

(56)

The first order condition is

$$(Gf_t)' - \gamma \Sigma x_t - 2 \Lambda x_t + \Lambda x_{t-1} = 0$$

(57)

with solution

$$x_t = (\gamma \Sigma - 2 \Lambda)^{-1} (Gf_t + \Lambda x_{t-1}).$$

(58)

A.4 The General Robust Position

Let $V$ be the value function, i.e., $V(x_{t-1}, f_t)$ is the optimal value of the agent’s problem given $x_{t-1}$ and $f_t$. The Bellman equation is then

$$V(x_{t-1}, f_t) = \max_{x_t} \min_{w_{t+1}} \left( x_t' \alpha_t - \frac{\gamma}{2} x_t' \Sigma x_t - \frac{1}{2} \Delta x_t' \Lambda \Delta x_t + \beta \theta w_{t+1}' w_{t+1} + \beta E_t(V(x_t, f_{t+1})) \right).$$

(59)

Guess that the value function takes the form

$$V(x_t, f_{t+1}) = -\frac{1}{2} x_t' A_{xx} x_t + x_t' A_{xf} f_{t+1} + \frac{1}{2} f_{t+1}' A_{ff} f_{t+1} + a_0$$

(60)
Solving for $w$ and the first order condition wrt. $w$.

The first order condition wrt. $w$ is

$$E_t(V(x_t, f_{t+1})) = -\frac{1}{2} x_t' A_{xx} x_t + x_t' A_{xf} E_t[(\Phi f_t + C w_{t+1} + C \epsilon_{t+1})]$$

$$+ \frac{1}{2} E_t[(\Phi f_t + C w_{t+1} + C \epsilon_{t+1})'] A_{ff} (\Phi f_t + C w_{t+1} + C \epsilon_{t+1}) + a_0$$

$$= -\frac{1}{2} x_t' A_{xx} x_t + x_t' A_{xf} \Phi f_t + x_t' A_{xf} C w_{t+1} + \frac{1}{2} f_t' \Phi' A_{ff} \Phi f_t$$

$$+ f_t' \Phi' A_{ff} C w_{t+1} + \frac{1}{2} w_{t+1}' C' A_{ff} C w_{t+1} + \frac{1}{2} E_t(\epsilon_{t+1}' C' A_{ff} C \epsilon_{t+1}) + a_0$$

The problem is then

$$\max_{x_t} \min_{w_{t+1}} (x_t' \alpha - \frac{\gamma}{2} x_t' \Sigma x_t - \frac{1}{2} \Delta x_t' \Lambda x_t + \beta \theta w_{t+1}' w_{t+1})$$

$$+ \beta [ -\frac{1}{2} x_t' A_{xx} x_t + x_t' A_{xf} \Phi f_t + x_t' A_{xf} C w_{t+1} + \frac{1}{2} f_t' \Phi' A_{ff} \Phi f_t$$

$$+ f_t' \Phi' A_{ff} C w_{t+1} + \frac{1}{2} w_{t+1}' C' A_{ff} C w_{t+1} + \frac{1}{2} E_t(\epsilon_{t+1}' C' A_{ff} C \epsilon_{t+1}) + a_0]$$

which becomes

$$x_t' \left( -\frac{1}{2} \beta A_{xx} - \frac{\gamma}{2} \Sigma - \frac{1}{2} \Lambda \right) x_t + x_t' (G f_t + \Lambda x_{t-1} + \beta A_{xf} \Phi f_t)$$

$$w_{t+1}' \left( \beta \theta I + \frac{1}{2} \beta C' A_{ff} C \right) w_{t+1} + w_{t+1}' \beta C' A_{ff} \Phi f_t$$

$$x_t' \beta A_{xf} C w_{t+1} - \frac{1}{2} x_{t-1}' \Lambda x_{t-1} + \frac{1}{2} f_t' \Phi' A_{ff} \Phi f_t + \frac{1}{2} E_t(\epsilon_{t+1}' C' A_{ff} C \epsilon_{t+1}) + \beta a_0$$

The first order condition wrt. $x_t$ is

$$-(\beta A_{xx} + \gamma \Sigma + \Lambda) x_t + G f_t + \Lambda x_{t-1} + \beta A_{xf} \Phi f_t + \beta A_{xf} C w_{t+1} = 0$$

and the first order condition wrt. $w_{t+1}$ is

$$2 \theta I + \beta C' A_{ff} C) w_{t+1} + \beta C' A_{ff} \Phi f_t + \beta C' A_{xf} x_t = 0.$$  

Solving for $w_{t+1}$ leads to

$$w_{t+1} = -(2 \theta I + C' A_{ff} C)^{-1}(C' A_{ff} \Phi f_t + C' A_{xf} x_t)$$

$$30$$
and substituting back into the first order condition for $x_t$ gives

$$
0 = - (\beta A_{xx} + \gamma \Sigma + \Lambda)x_t + Gf_t + \Lambda x_{t-1} + \beta A_{xf} \Phi f_t
- \beta A_{xf} C (2\theta I + C'A_{ff} C)^{-1} (C'A_{ff} \Phi f_t + C'A'_{xf} x_t) = 0
$$

Solving gives the robust position $x_t$

$$
x_t = (\beta A_{xx} + \gamma \Sigma + \Lambda + \beta A_{xf} C (2\theta I + C'A_{ff} C)^{-1} C'A_{xf})^{-1}
(Gf_t + \Lambda x_{t-1} + \beta A_{xf} \Phi f_t - \beta A_{xf} C (2\theta I + C'A_{ff} C)^{-1} C'A_{xf} \Phi f_t).
$$

### B The Aim and Target Portfolios

The Bellman equation is

$$
- \frac{1}{2} x'_{t-1} A_{xx} x_{t-1} + f'_t A_{fx} x_{t-1} + \frac{1}{2} f''_t A_{ff} f_t + A_0
= x'_t Gf_t - \frac{\gamma}{2} x'_t \Sigma x_t - \frac{1}{2} (x_t - x_{t-1})' \Lambda (x_t - x_{t-1}) + \beta \theta w'_{t+1} w_{t+1} + \beta E_t(V(x_t, f_{t+1}))
= x'_t Gf_t - \frac{\gamma}{2} x'_t \Sigma x_t - \frac{1}{2} (x_t - x_{t-1})' \Lambda (x_t - x_{t-1}) + \beta \theta (K'_f f_t + K_x x_{t-1})' (K'_f f_t + K_x x_{t-1})
+ \frac{1}{2} \beta x'_t A_{xx} x_t + \beta x'_t A_{xf} ((\Phi + CK)' f_t + CK_x x_{t-1})
+ \frac{1}{2} \beta ((\Phi + CK)' f_t + CK_x x_{t-1})' A_{ff} ((\Phi + CK)' f_t + CK_x x_{t-1}) + \frac{1}{2} \beta E_t (\epsilon'_{t+1} C'A_{ff} C \epsilon_{t+1})
$$

Differentiate the Bellman equation wrt. $x_{t-1}$ (using the envelope theorem to keep $x_t$ and fixed) gives

$$
A_{xx} x_{t-1} + A_{xf} f_t = \Lambda (x_t - x_{t-1}) + 2 \beta \theta K'_x (K'_f f_t + K_x x_{t-1}) + \beta K'_x C'A'_{xf} x_t
- \beta K'_x C'A_{ff} ((\Phi + CK)' f_t + CK_x x_{t-1})
$$

such that

$$
(A - \beta K'_x C'A'_{xf}) x_t = \Lambda x_{t-1} - A_{xx} x_{t-1} + A_{xf} f_t + 2 \beta \theta K'_x (K'_f f_t + K_x x_{t-1}) + \beta K'_x C'A_{ff} ((\Phi + CK)' f_t + CK_x x_{t-1})
$$
and thus
\[
x_t = (\Lambda - \beta K_x'C' A'_{xf})^{-1} \left( \Lambda x_{t-1} - A_{xx} x_{t-1} + A_{xf} f_t \right) \\
+ 2\beta \theta K'_x (K_f f_t + K_x x_{t-1}) + \beta K'_x C' A'_{ff} \left( (\Phi + CK_f) f_1 + CK_x x_{t-1} \right) \\
= x_{t-1} + (\Lambda - \beta K'_x C' A'_{xf})^{-1} \left( \beta K'_x C' A'_{xf} x_{t-1} - A_{xx} x_{t-1} + A_{xf} f_t \right) \\
+ 2\beta \theta K'_x (K_f f_t + K_x x_{t-1}) + \beta K'_x C' A'_{ff} \left( (\Phi + CK_f) f_1 + CK_x x_{t-1} \right) \\
= x_{t-1} + (\Lambda - \beta K'_x C' A'_{xf})^{-1} A_{xx} (\text{aim}_t - x_{t-1})
\]

where \( \text{aim}_t = \text{aim}_f f_t + \text{aim}_x x_{t-1} \) and (using the recursion for \( A_{xf} \))

\[
\text{aim}_f = A_{xx}^{-1} (A_{xf} + 2\beta \theta K'_x K_f + \beta K'_x C' A'_{ff} (\Phi + CK_f)) \\
= A_{xx}^{-1} (J'_x J^{-1} (G + \beta A_{xf} (\Phi + CK_f)) + 2 N_{A_{xf}}) \\
\text{aim}_x = A_{xx}^{-1} (\beta K'_x C' A'_{xf} + 2\beta \theta K'_x K_x + \beta K'_x C' A'_{ff} CK_x)
\]

Next, I show that \( \text{aim}_f \) can be written as a weighted average of future target portfolios.

To write the aim portfolio as a weighted average of the future target portfolios, define the target portfolio as

\[
T_t = \text{Markowitz}_t \\
T_{t+s} = E^w_t (\text{Markowitz}_{t+s}) + (\gamma \Sigma Z)^{-1} N_{A_{xf}} E^w_t (f_{t+s})
\]

where \( Z = J'_x J^{-1} \) and \( N_{A_{xf}} \) is defined in (54). \( E^w \) indicates that the expectation is taken under the worst-case model where the mean-reversion is distorted by the evil agent: \( f_{t+1} = (\Phi + CK_1) f_t + C \varepsilon_{t+1} \).

That is, the current target is the current Markowitz portfolio, but the expected future targets are distorted relative to the expected Markowitz portfolio.
Using the recursion for $A_{xf}$ gives

\[
aim_{f}f_{t} = A_{xx}^{-1}J'_{x}J^{-1}(\gamma \Sigma \times \text{Markowitz}_{t} + \beta A_{xf}(\Phi + CK_{f})f_{t}) + A_{xx}^{-1}2N_{A_{xf}}f_{t} \\
= A_{xx}^{-1}J'_{x}J^{-1}\left(\gamma \Sigma \times \text{Markowitz}_{t} + \beta \{J'_{x}J^{-1}(G + \beta A_{xf}(\Phi + CK_{f})) + N_{A_{xf}}\}E_{w}^{w}(f_{t+1})\right) \\
+ A_{xx}^{-1}2N_{A_{xf}}f_{t} \\
= A_{xx}^{-1}J'_{x}J^{-1}\left(\gamma \Sigma \times \text{Markowitz}_{t} + \beta J'_{x}J^{-1}\gamma \Sigma \times E_{w}^{w}(\text{Markowitz}_{t+1}) \\
+ \beta^2 J'_{x}J^{-1}A_{xf}(\Phi - CK_{f})E_{w}^{w}(f_{t+1}) + \beta N_{A_{xf}}E_{w}^{w}(f_{t+1})\right) + A_{xx}^{-1}2N_{A_{xf}}f_{t}
\]

and repeating the substitution for $A_{xf}$ leads to

\[
aim_{f}f_{t} = A_{xx}^{-1}\gamma \Sigma \sum_{s=0}^{\infty} \beta^{s}Z^{s} \times \text{Target}_{t+s} + A_{xx}^{-1}2N_{A_{xf}}f_{t} \tag{92}
\]

where, as defined above, $Z = J'_{x}J^{-1}$. 

33
References


Figure 1: Effect of Robustness

The figure illustrates how the position changes when the desire for robustness is increased. There are two assets, and the expected return on each is described by its own predictor. The predictor is slowly mean-reverting for asset 1 and quickly mean-reverting for asset 2. The myopic Markowitz positions converge to zero over time, but converge more quickly for the second asset. Therefore, the investor wants to take a larger position in asset 1, and without robustness the investor trades towards the aim portfolio. As the desire for robustness is increased, the robust position moves from the optional position toward the super-robust position.
The figure illustrates how the target- and aim positions change with robustness. There are two assets, and the expected return on each is described by one of two predictors. The predictor is slowly mean-reverting for asset 1 and quickly mean-reverting for asset 2. The myopic Markowitz positions converge to zero over time, but converge more quickly for the second asset. Therefore, the investor wants to take a larger position in asset 1, and without robustness the investor trades toward the aim portfolio. With robustness, the target portfolios are changed to reflect the mean-reverting dynamics under the worst-case model. Since the evil agent distorts the signal for the attractive asset 1 the most, the new target portfolios lie closer to the diagonal. The robust aim portfolio is given in Proposition 4.2 and 4.3.
Figure 3: Robustness and Predictor Innovation Variance
The figure illustrates how the target- and aim positions change with robustness. There are two assets, and the expected return on each is described by one of two predictors. The predictor for asset 1 has innovations with a high variance, and the predictor for asset 2 has innovations with a low variance. The myopic Markowitz positions converge to zero over time, equally fast for both assets, and the optimal aim portfolio thus has an equal position in both assets. With robustness, the target portfolios are changed to reflect the mean-reverting dynamics under the worst-case model. Since the evil agent distorts the signal for asset 1 the most, the new target portfolios have a smaller position in asset 1. The robust aim portfolio is given in Proposition 4.2 and 4.3.
Figure 4: Cumulative Returns

The figure shows the cumulative monthly excess returns for the myopic Markowitz strategy as well as for the robust strategies. The results for the myopic Markowitz strategy are before transaction costs, but the results for the robust strategies are after transaction costs. All strategies are scaled to an annualized volatility of 10%.
Figure 5: All Asset Classes

The figure shows the cumulative monthly excess returns and draw downs for the robust strategy as well as the rank-weighted momentum strategy, based on all asset classes. First, I construct momentum portfolios for each asset class and scale each to an annualized volatility of 10%. Second, I form an equal-weighted portfolio of the four strategies and scale that to have an annualized volatility of 10%. The results for the robust strategies are after transaction costs, whereas the results for the rank weighted strategy are before transaction costs.
Table 1: Performance Statistics: \( \alpha \) and \( \beta \)

| Panel A: Markowitz and Rank-Weighted Strategies, scaled to 10% volatility |
|-----------------------------|-----------------------------|-----------------------------|
| Data Set                    | Markowitz, No TC            | Rank, No TC                 |
|                             | SR  | \( \alpha \) | \( \beta \) | SR  | \( \alpha \) | \( \beta \) | SR  | \( \alpha \) | \( \beta \) |
| 10 Industries               | 0.62 | 5.88** | 0.05 | 0.35 | 4.02** | -0.09 |
| 18 Countries                | 0.61 | 5.62** | 0.10 | 0.39 | 3.92** | -0.01 |
| 24 Commodities              | 0.96 | 9.53** | 0.06 | 0.46 | 4.50** | 0.11 |
| 9 FX Rates                  | 0.80 | 7.96** | 0.03 | 0.21 | 2.26 | -0.05 |
| All Asset Classes           | 1.18 | 11.43** | 0.08 | 0.56 | 5.48** | 0.03 |

| Panel B: Robust Strategy, robustness is scaled to 10% volatility |
|-----------------------------|-----------------------------|-----------------------------|
| Data Set                    | High TC                     | Medium TC                   | Low TC             |
|                             | SR  | \( \alpha \) | \( \beta \) | SR  | \( \alpha \) | \( \beta \) | SR  | \( \alpha \) | \( \beta \) |
| 10 Industries               | 0.40 | 2.98* | 0.17*** | 0.51 | 4.50** | 0.10 | 0.62 | 5.72** | 0.08 |
| 18 Countries                | 0.45 | 3.30* | 0.25**  | 0.66 | 6.04** | 0.12 | 0.74 | 6.86** | 0.11 |
| 24 Commodities              | 0.43 | 4.26** | 0.04   | 0.84 | 8.37***| 0.07 | 0.95 | 9.43***| 0.07 |
| 9 FX Rates                  | 0.49 | 4.47** | 0.19** | 0.81 | 8.03***| 0.03 | 0.83 | 8.17***| 0.04 |
| All Asset Classes           | 0.72 | 6.10***| 0.24*** | 1.11 | 10.46***| 0.13 | 1.28 | 12.44***| 0.09 |

| Panel C: Optimal Strategy, \( G \) is scaled to 10% volatility |
|-----------------------------|-----------------------------|-----------------------------|
| Data Set                    | High TC                     | Medium TC                   | Low TC             |
|                             | SR  | \( \alpha \) | \( \beta \) | SR  | \( \alpha \) | \( \beta \) | SR  | \( \alpha \) | \( \beta \) |
| 10 Industries               | 0.35 | 1.67 | 0.32*** | 0.59 | 5.21*** | 0.12* | 0.59 | 5.46*** | 0.08 |
| 18 Countries                | 0.35 | 1.63 | 0.39*** | 0.64 | 5.69*** | 0.15* | 0.67 | 6.22*** | 0.10 |
| 24 Commodities              | 0.33 | 3.37** | -0.03 | 0.83 | 8.29*** | 0.02 | 0.99 | 9.90*** | 0.05 |
| 9 FX Rates                  | 0.38 | 3.22 | 0.25*** | 0.80 | 7.75*** | 0.10 | 0.84 | 8.33*** | 0.03 |
| All Asset Classes           | 0.58 | 4.26** | 0.34*** | 1.15 | 10.83***| 0.15* | 1.24 | 12.04***| 0.09 |

| Panel D: Optimal Strategy, \( \gamma \) is scaled to 10% volatility |
|-----------------------------|-----------------------------|-----------------------------|
| Data Set                    | High TC                     | Medium TC                   | Low TC             |
|                             | SR  | \( \alpha \) | \( \beta \) | SR  | \( \alpha \) | \( \beta \) | SR  | \( \alpha \) | \( \beta \) |
| 10 Industries               | 0.38 | 2.30 | 0.25**  | 0.57 | 5.12*** | 0.10 | 0.60 | 5.59*** | 0.07 |
| 18 Countries                | 0.37 | 2.22 | 0.31*** | 0.68 | 6.17*** | 0.12 | 0.64 | 5.96*** | 0.09 |
| 24 Commodities              | 0.37 | 3.72** | -0.01 | 0.90 | 8.97*** | 0.03 | 1.00 | 9.94*** | 0.05 |
| 9 FX Rates                  | 0.44 | 3.85** | 0.24** | 0.83 | 8.11*** | 0.08 | 0.83 | 8.26*** | 0.03 |
| All Asset Classes           | 0.64 | 4.98***| 0.30*** | 1.19 | 11.37***| 0.12 | 1.24 | 11.99***| 0.09 |

| Panel E: Optimal Strategy, \( \Phi \) is scaled to 10% volatility |
|-----------------------------|-----------------------------|-----------------------------|
| Data Set                    | High TC                     | Medium TC                   | Low TC             |
|                             | SR  | \( \alpha \) | \( \beta \) | SR  | \( \alpha \) | \( \beta \) | SR  | \( \alpha \) | \( \beta \) |
| 10 Industries               | 0.37 | 1.81 | 0.32*** | 0.60 | 5.22*** | 0.12* | 0.58 | 5.35*** | 0.07 |
| 18 Countries                | 0.41 | 2.09 | 0.40*** | 0.65 | 5.72*** | 0.15* | 0.66 | 6.02*** | 0.11 |
| 24 Commodities              | 0.36 | 3.59** | -0.02 | 0.83 | 8.29*** | 0.02 | 0.97 | 9.65*** | 0.07 |
| 9 FX Rates                  | 0.39 | 3.23* | 0.27** | 0.79 | 7.67*** | 0.09 | 0.83 | 8.26*** | 0.03 |
| All Asset Classes           | 0.62 | 4.55***| 0.36*** | 1.15 | 10.79***| 0.15* | 1.21 | 11.66***| 0.10 |

***, **, and * indicate significance at the 1%, 5%, and 10% levels, respectively.